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Translated by N.H.C.

UDC 62-50

PROGRAM AND POSITION ABSORPTION IN DIFFERENTIAL GAMES

PMM Vol. 36, №4, 1972, pp. 740-743

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(Received October 25, 1971)

We investigate the relations between the position and the program absorption sets. We cite an example in which the construction of the position absorption set [1 – 3] is reduced to the determination of a finite number of program absorption sets [4, 5].

It is known that in the general case the construction of a position absorption set leads to the determination of a countable sequence of program absorption sets [1, 3, 6, 7]. Also well known are the cases when the position absorption set is determined by one program absorption operation [2, 3, 5, 8]. We consider a linear differential game of pursuit. Let the motion of a conflict-controlled system be described by the equation

$$dx / dt = A(t)x + u - v \tag{1}$$

Here x is the n-dimensional system phase vector; $A(t)$ is an $n \times n$ matrix with coefficient depending continuously on t ; u and v are the controls of the first and second players, respectively, whose realizations are constrained by $u[t] \in P_t$, $v[t] \in Q_t$, where the closed convex sets P_t and Q_t depend piecewise-continuously on t.

In the phase space R_n we are given a set M which is usually assumed closed and convex. The solution of the pursuit problem consists of having to construct the first player's strategy which guarantees that the phase point $x[t]$ is taken onto the aim set M. It is assumed that information on the game position $(t, x[t])$ realized is available to the pursuer. Thus, the pursuit strategies are certain functions $U = U(t, x)$. The classes of players' strategies, containing the solution of the position differential game, were introduced in [2, 7].

Let us briefly describe certain elements of extremal construction used in solving position differential games. Let a certain closed set $W(t) \subset R_n$ be associated with each value of t $(t_0 \leqslant t \leqslant \vartheta)$. We assume that the system of sets $W(t)$ possesses the property of strong u-stability [2, 7], moreover, $W(\vartheta) \subset M$, $x_0 \in W(t_0)$. Then we can construct a strategy $U_e = U_e(t, x)$, extremal to the system of sets $W(t)$, which guarantees the termination of the pursuit game at the instant $t = \vartheta$. From the definition of the extremal strategy [2, 5, 7] we see that when the system of stable sets is known, the construction of the corresponding extremal strategy does not present great difficulty. Thus, the basic problem is to construct the sets $W(t)$ $(t_0 \leqslant t \leqslant \vartheta)$, possessing the properties listed above.

We turn to an examination of the results concerning the problem of constructing the desired strongly u-stable sets $W(t)$ $(t_0 \leqslant t \leqslant \vartheta)$. The basic concept used in the investigation of this problem is that of a program absorption set $W_1(t, \vartheta; M)$ [4, 5]. Conditions have been stated in [3, 5, 8] under which the system of sets $W_1(t, \vartheta; M)$ $(t_0 \leqslant t \leqslant \vartheta)$ possesses the property of strong u-stability. The equality $W_1(\vartheta, \vartheta; M) = M$, follows from the definition of the set $W_1(t, \vartheta; M)$, therefore, in the given case, and under the condition $x_0 \in W_1(t_0, \vartheta; M)$, the strategy $U_e = U_e(t, x)$, extremal to the system of program absorption sets, guarantees the termination of the pursuit game at the instant $t = \vartheta$. We note that the construction of the program absorption sets is a comparatively simple task (the relations defining these sets have been presented in [4, 5, 8]. Therefore, the pursuit problem is, in effect, solved in the given case.

In the general case the program absorption sets do not possess the stability property, therefore, to construct stable sets we have to introduce the notion of position absorption [2, 7]. With the aid of the position absorption sets $W_0(t, \vartheta; M)$ we can formulate the necessary and sufficient condition for game termination at the instant $t = \vartheta$; the fulfillment of the inclusion $x_0 \in W_0(t_0, \vartheta; M)$ serves as such a condition. Thus, the construction of the position absorption sets $W_0(t, \vartheta; M)$ comprises one of the fundamental stages in the investigation of differential games. It is known that the sets $W_0(t, \vartheta; M)$, specified by the property of position absorption [2, 7], can be determined also by means of a retrograde procedure [1, 3], and, moreover, the basic element of this procedure turns out to be the program absorption operation.

As we have already noted above, the construction of program absorption sets is a comparatively simple task. However, in the general case, to construct a position absorption set by the retrograde procedure we have to introduce a countable sequence of program absorption operations. In the study of the conditions for an effective construction of the sets $W_0(t, \vartheta; M)$ the question arises of the existence of situations when $W_0(t, \vartheta; M) \neq W_1(t, \vartheta; M)$ but to construct the set $W_0(t, \vartheta; M)$ we need to carry out only a finite number of program absorption operations. This question has an affirmative answer. We present an appropriate example below. As a preliminary we note that in this example the position absorption set is determined by two program absorption operations and here we obtain the following relation:

$$W_* \neq W_0 \neq W_1, \qquad W_* \subset W_0 \subset W_1$$

where W_* is the absorption set determined by the construction in [9]. We note that the sets W_* can be defined as the sets of regular program absorption [10].

We procee to the example. Let the motion of a two-dimensional phase vector $x[t] = (x_1[t], x_2[t])$ be described by Eq. (1) with $A(t) \equiv 0$. The constraints on the controls have the following forms:

$$u[t] \in (1+\sqrt{2})G_1 \quad \text{for } t \in [0, 1]$$
$$u[t] \in (1+\sqrt{2})G_2 \quad \text{for } t > 1$$
$$v[t] \in G_2 \quad \text{for } t \in [0, 1], \quad v[t] \in G_1 \quad \text{for } t > 1$$

where the sets G_1 and G_2 are the squares

$$G_1 = \{g: |g_1| \leqslant 1, \quad |g_2| \leqslant 1\}$$
$$G_2 = \{g: |g_1| + |g_2| \leqslant 1\}$$

The set M consists of a single point – the origin (0, 0). We note that in this example the sets which restrict the choice of the players' controls depend piecewise-continuously on time, and this dependency has a discontinuity at the instanr $t = 1$. The given example can be altered so that the constraints on the controls are constant, while the system of equations describing the motion of the phase vector $x[t]$ is nonstationary with coefficients which depend continuously on time. Here the fundamental property of the given example is preserved.

We describe the construction of the set $W_0(0, 2; M)$, i.e. we take $t = 0$ as the initial time and $\vartheta = 2$ as the final time. Let us examine the program absorption set $W_1(t, 2; M)$ for $t \in [1, 2]$. On this interval the constraints on the controls are time-independent and the matrix $A(t) \equiv 0$, therefore, it is not difficult to show that the set $W_1(t, 2; M)$ is given by the relation

$$W_1(t, 2; M) = [(2-t)(1+\sqrt{2})G_2 + M] \overset{*}{-} (2-t)G_1, \quad t \in [1, 2]$$

Here $B + C$ denotes the collection of all points of the form $b + c$ $(b \in B,\ c \in C)$ and the symbol $\overset{*}{-}$ denotes geometric subtraction, i.e. $D = B \overset{*}{-} C = \{d : d + C \subset B\}$. Taking into account that the set M consists of the single point (0, 0), we obtain

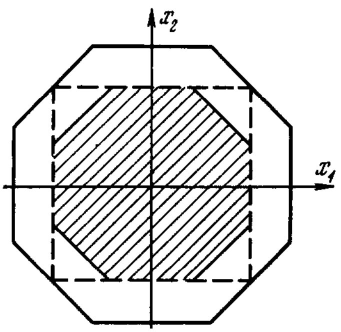

Fig. 1

$$W_1(t, 2; M) = (2-t)H_1, \; H_1 = \{c : |c_1| + |c_2| \leqslant 2\}$$

It is directly verified that the system of sets $W_1(t, 2; M)$ $(1 \leqslant t \leqslant 2)$ is strongly u-stable. It is known that the equality $W_1(t, 2; M) = W_0(t, 2; M)$ follows from this property [7, 8]. In particular, for $t = 1$ we obtain

$$W_0(1, 2; M) = H_1 \tag{2}$$

Let us now examine the set $W_0(t, 2; M)$ $(0 \leqslant t \leqslant 1)$. Having constructed the program absorption sets $W_1(t, 1; W_0(1, 2; M))$ $(0 \leqslant t \leqslant 1)$, we have

$$W_1(t, 1; W_0(1, 2; M)) = [(1+\sqrt{2})(1-t)G_1 + W_0(1, 2; M)] \overset{*}{-} (1-t)G_2 \tag{3}$$

The sets $W_1(t, 2; W_0(1, 2; M))$ $(0 \leqslant t \leqslant 1)$ also are strongly u-stable, therefore, the equality

$$W_0(t, 2; M) = W_1(t, 1; W_0(1, 2; M)) \tag{4}$$

is valid. From relations (2) – (4) we can get that for $t = 0$ the set $W_0(t, 2; M)$ is a square with side $l_1 = 2 + 2\sqrt{2}$. We compare the set $W_0(0, 2; M)$ with the set

$$W_1(0, 2; M) = [(1+\sqrt{2})G_1 + (1+\sqrt{2})G_2] \overset{*}{-} (G_1 + G_2)$$

It is a regular octagon with side $l_2 = 2\sqrt{2}$.

Finally, we determine the set $W_*(0, 2)$ by the approach suggested in [9]. We obtain

$$W_*(0,2) = [(1+\sqrt{2})\,G_1 \overset{*}{-} G_2] + [(1+\sqrt{2})\,G_2 \overset{*}{-} G_1]$$

The set $W_*(0, 2)$ turns out to be an octagon with side $l_8 = 2$. The program absorption set $W_1(0, 2; M)$ contains the set $W_0(0, 2; M)$ which, in turn, contains the set $W_*(0, 2)$. In Fig.1 the boundary of set $W_1(0, 2; M)$ is shown by a solid line, the boundary of set $W_0(0, 2; M)$ by a dashed line, while set $W_*(0, 2)$ is shown cross-hatched. Thus in the given example the position absorption set is determined by two program absorption operations.

The author thanks N.N.Krasovskii for posing the problem and for discussing the work.

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Translated by N.H.C.